On Quotient Spaces *

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Quotient spaces are examined as they occur in canonical formalisms with constraints. Illustrative examples are taken from classical mechanics, from general-relativistic theories, and from the theory of the electron microscope.

In the Hamiltonian formulation of general relativity and similar theories there arises a curious hierarchy of symplectic manifolds. The canonical variables span a function space that has been called the extended phase space. Imbedded within the extended phase space there is the constraint hypersurface defined by Dirac's Hamiltonian constraints [1]. Within the constraint hypersurface there are defined subspaces referred to as equivalence classes; each of these classes represents one solution of Einstein's field equations in terms of all possible choices of Cauchy data on space-like three-dimensional surfaces. An finally, the quotient space of the constraint hypersurface with respect to the equivalence classes is known as the reduced phase space [2].

Quotient spaces with respect to different subspaces have been constructed, for instance, to provide an invariant basis for phase integral quantization [3], or in order to avoid some of the implications of a "frozen formalism" [4]. All these constructions have in common that the equivalence classes are the orbits of a selected group of transformations, and that there exists a homomorphism between a larger group and its factor group with respect to the selected group. This relationship appears in some respects more fundamental than the technology of a Hamiltonian formalism; in any case, the construction of the quotient space and the group homomorphism tend to illuminate each other.

Intuitively, with any set of physical situations, the construction of equivalence classes amounts to the collection of certain sets that have "something in common". What they have in common may be the intrinsic physical characteristics of the situation, regardless of the mode of description, or more, or less. Once equivalence classes have been formed, the reversible one-to-one mappings of a class on itself represent a transformation group. If there are many equivalence classes, the tensor product of all these groups forms a much larger group, which shall be referred to as the (maximal) invariance group. Any element of this large group maps each equivalence class on itself [5].

Incidentally, equivalence classes need not all have the same dimensionality. In general relativity, for instance, if all representations of one space-time be considered an equivalence class, then space-times with isometry groups have smaller equivalence classes than others.

We can now construct a group of mappings of all physical situations under consideration on each other of which the invariance group forms a normal subgroup. Begin with a set of reversible one-to-one mappings, **M**, which map any two situations belonging to the same equivalence class into one new equivalence class, that is to say, each **M** maps the equivalence classes intact on each other. If these mappings **M** do not form a group to begin with, a group may be constructed by the usual closure operation: Reciprocals and products of mappings are added indefinitely until the set of mappings is closed under both operations. Call this new group $\bar{\mathbf{M}}$; the invariance group $\bar{\mathbf{G}}$ will then be a normal subgroup of $\bar{\mathbf{M}}$. The factor group $\mathbf{M}/\bar{\mathbf{G}}$ will then have a homomorphism with $\bar{\mathbf{M}}$. The corresponding quotient space will consist of all physical situations distinct with respect to the chosen equivalence classes. Its elements will not be the original physical situations, but the equivalence classes. The elements of $\bar{\mathbf{M}}/\bar{\mathbf{G}}$ will map that quotient space on itself.

Conventionally, quotient spaces have been constructed within a canonical (Hamiltonian) for-

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malism from a different point of departure than the one outlined here. Starting with a hypersurface imbedded in the extended phase space, the constraint hypersurface, one begins by asking for all those canonical mappings of the extended phase space on itself that also map the constraint hypersurface on itself. The generators of these (infinitesimal) mappings are the first-class variables of Dirac, and these form a Lie algebra. Among the first-class variables are a subset, the first-class constraints, i. e. those variables that vanish on the constraint hypersurface. The first-class constraints not only form a Lie algebra of their own, this latter Lie algebra is an invariant subalgebra of the former; that is why there exists a quotient algebra and a homomorphism. The equivalence classes simply are the orbits of the normal subgroup of the group of canonical mappings of the constraint hypersurface on itself whose generators vanish.

Thus, if we begin with the canonical mappings of the constraint hypersurface on itself, the existence of a normal subgroup, and of its orbit, is automatic. Outside canonical mapping theory, this is not generally true. Given any transformation group, we always can introduce a subgroup of those mappings in which arbitrarily selected subspaces ("equivalence classes") are mapped on themselves, but this subgroup will not be invariant. For this to happen in the canonical theory, it was essential that the set of constraints among the first-class variables be nonempty, and that the equivalence classes be defined as the orbits of the first-class constraints. The invariance of the subgroup then brings it about that all first-class variables map the equivalence classes intact on each other.

The appearance of first-class constraints, in turn, is characteristic of theories whose invariance groups are infinite-dimensional. If the invariance groups are finite-dimensional (Lie groups), no first-class constraints exist.

I shall discuss a few examples. In Dirac's version of general relativity the canonical field variables, g_{mn}, p^{mn}, are constrained at each point by the Hamiltonian constraints, $\mathcal{H}_s = 0$, $\mathcal{H}_L = 0$. These constraints are the generators of the infinitesimal curvilinear coordinate transformations; their Lie algebra is isomorphic with the infinitesimal invariance group of the theory [5]. Accordingly, the natural equivalence classes of Dirac's formalism are the sets of (permissible) fields of g_{mn}, p^{mn} that are accessible to each other by canonical transformations obtained from the exponentiation of those generated by the Hamiltonian constraints. Infinitesimal mappings of equivalence classes on each other are provided by so-called first-class variables, i. e. by functionals having zero brackets with all of Dirac's constraints. Such functionals are invariants under coordinate transformations and, by implication, also constants of the motion. The first-class variables form a Lie algebra (under Poisson brackets), and the first-class constraints an invariant subalgebra. The set of all permissible sets of g_{mn}, p^{mn} form the constraint hypersurface of the theory, and the elements of the quotient space are the intrinsically distinct solutions of Einstein's field equation, regardless of coordinatization.

This example is based on the adoption of the four-dimensional coordinate transformations as the invariance group. It leads to a quotient space whose canonical mappings are generated by constants of the motion, analogous to the phase space contemplated in Hamilton-Jacobi theory.

One can retain a semblance of dynamics by changing the invariance group, restricting it to three-dimensional coordinate transformations, i. e. transformations that map the Cauchy hypersurface on itself. One equivalence class, a 3-geometry, then corresponds to an element of Wheeler's superspace [4]. The canonical transformations mapping one 3-geometry on another 3-geometry form, of course, a much larger group than those that map one Ricci-flat manifold on another Ricci-flat manifold. Their generators are required to have vanishing Poisson brackets with the $\mathcal{H}_s$, but not with $\mathcal{H}_L$. As a result, the latter constraint changes its form under the transformations considered. But conversely, the generators mapping 3-geometries on 3-geometries are not constants of the motion; they propagate in accordance with their Poisson brackets with $\mathcal{H}_L$. This formalism reduces to the one described previously if one applies to it a Hamilton-Jacobi transformation.

Another modification, in the direction opposite to that leading to the superspace, has been proposed by the author in order to obtain phase integrals [3]. If instead of all possible solutions of the field equations one considers a subset, characterized by the fixation of the numerical values of some selected, mutually commuting constants of the motion, $A = a_0$, these fixations may be considered as additional first-class constraints. To the extent that these new variables A map distinct equivalence classes on each other, these equivalence classes may now be considered "accessible" to each other, and combined into "superclasses". In classical mechanics, and with suitably selected variables A, this procedure can lead to superclasses that are multiply-connected manifolds. Phase integrals are integrals over non-contractible closed paths, which then serve as a basis for quantization.

The method of superclasses has one application in classical mechanics that provides a very intuitive example. Consider an instrument using particle rays, such as an electron microscope. The particle trajectories in such an instrument obey equations of motion of the form

$$\dot{q}_k = \partial H/\partial p_k\,, \quad \dot{p}_k = -\partial H/\partial q_k\,, \quad k=1,\ldots,3\,.$$

It is well known that by parametrizing these trajectories one can put the time coordinate t on the same basis as the other configuration coordinates, resulting in new equations of motion, of the form

$$\mathrm{d}q_k/\mathrm{d}\theta = \partial\bar{H}/\partial p_k\,, \quad \mathrm{d}p_k/\mathrm{d}\theta = -\partial\bar{H}/\partial q_k\,, \quad k=1,\ldots,4\,.$$

$$q_4 \equiv t\,, \quad p_4 = -H\,,$$

$$\bar{H} \equiv H + p_4 = 0\,.$$

$\bar{H}$ is the Hamiltonian constraint, and the equivalence classes are simply the trajectories on the constraint hypersurface, which is coordinated by $q_1, \ldots, q_4, p_1, \ldots, p_3$. Two trajectories that differ from each only by having been started at different times represent distinct equivalence classes.

In order to treat an electron microscope as an optical system one must eliminate the time coordinate t. The instrument will operate only if the entering particles are all at the same energy (or in a narrow energy range). Suppose, then, that we add a second constraint, $H=E$, with E being some particular chosen operating energy. $H-E$ has a vanishing Poisson bracket with H, being a constant of the motion, and it generates displacement of all trajectories along the time axis. The two constraints together generate superclasses, and each superclass consists of all the trajectories that differ from each other only with respect to the time at which a particle enters the instrument. Whereas the original extended phase space is eight-dimensional, being the product of the six-dimensional particle phase space $(\boldsymbol{x}, \boldsymbol{p})$, the time axis, and the (negative) energy axis, the two constraints together restrict the constraint hypersurface to a six-dimensional manifold, containing only those points in which $-p_4$ equals the energy and in which the energy, in turn, is restricted to the chosen numerical value E. The superclasses are two-dimensional, hence, the quotient space four-dimensional.

The reduction of the problem to one manifestly belonging to ray optics is accomplished by means of the second (not the Hamiltonian) constraint. We set up preliminary classes of points that are mapped on each other by the constraint $p_4 + E = 0$. These are all the points on the constraint hypersurface differing from each other only with respect to the value of t. As long as the Hamiltonian constraint is disregarded, the constraint hypersurface will be seven-dimensional, corresponding to the product of the ordinary six-dimensional phase space $(\boldsymbol{x}, \boldsymbol{p})$ by the time axis. The preliminary quotient space will be six-dimensional, and homeomorphic to the phase space $(\boldsymbol{x}, \boldsymbol{p})$. It differs from the latter in terms of the dynamics.

The dynamics is provided by the Hamiltonian constraint, which now takes the form

$$\bar{H}(\boldsymbol{x}, \boldsymbol{p}) = H(\boldsymbol{x}, \boldsymbol{p}) - E = 0\,.$$

Ray trajectories will be generated if this constraint is multiplied by an arbitrary positive-definite, non-vanishing factor and then used as the Hamiltonian. The choice of multiplier is equivalent to the choice of parameter for the trajectories, and this parameter need not be the time. It might, for instance, be the path length.

The eikonal equation (i. e., the Hamilton-Jacobi equation of ray optics) takes the form

$$H(\boldsymbol{x}, \nabla S) - E = 0\,.$$

It is well known that the same equation is obtained as the lowest approximation if the Schrödinger equation is subjected to a WBK(J) approximation procedure. The next approximation would yield elementary diffraction effects, etc.

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